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IDEAL EINSTEIN, CONFORMALLY FLAT AND SEMI-SYMMETRIC IMMERSIONS

BY

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ABSTRACT

Recently, B. Y. Chen introduced a new intrinsic invariant of a manifold, and proved that every n-dimensional submanifold of real space forms $R^m(\varepsilon)$ of constant sectional curvature ε satisfies a basic inequality $\delta(n_1,\dots,n_k) \le c(n_1,\dots,n_k)H^2 + b(n_1,\dots,n_k)\varepsilon$, where H is the mean curvature of the immersion, and $c(n_1,\dots,n_k)$ and $b(n_1,\dots,n_k)$ are constants depending only on $n_1,\dots,n_k,n$ and k. The immersion is called **ideal** if it satisfies the equality case of the above inequality identically for some k-tuple $(n_1,\dots,n_k)$. In this paper, we first prove that every ideal Einstein immersion satisfying $n \ge n_1+\cdots+n_k+1$ is totally geodesic, and that every ideal conformally flat immersion satisfying $n \ge n_1+\cdots+n_k+2$ and $k \ge 2$ is also totally geodesic. Secondly we completely classify all ideal semi-symmetric hypersurfaces in real space forms.

1. Introduction and main theorems

Let M be an n-dimensional Riemannian manifold. Denote by $k(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p(M)$, $p \in M$. For any

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orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_p(M)$, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i \neq j} K(e_i \wedge e_j), \quad 1 \leq i, j \leq n.$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{\alpha \neq \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For any integer $k \geq 0$, denote by $\Psi(n, k)$ the finite set consisting of unordered k-tuples $(n_1, \ldots, n_k)$ of integers $n_i \geq 2$, satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Then let $\Psi(n)$ be the union $\bigcup_{k \geq 0} \Psi(n, k)$.

For each k-tuple $(n_1, \ldots, n_k) \in \Psi(n)$, B. Y. Chen introduced in [1, 2, 3] a Riemannian invariant $\delta(n_1, \ldots, n_k)$ by

$$2\delta(n_1, \ldots, n_k) = \tau - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{1.1}$$

where at each point $p \in M^n, L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$. And B. Y. Chen also proved in [3] the following optimal relationship between the invariants $\delta(n_1, \ldots, n_k)$ and the squared mean curvature H^2 for an arbitrary submanifold in a real space form.

THEOREM A: *Let M^n be an n-dimensional submanifold in a real space form $R^m(\varepsilon)$ of constant curvature ε. Then for each k-tuple $(n_1, \ldots, n_k) \in \Psi(n)$, we have*

$$\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)H^2 + b(n_1, \ldots, n_k)\varepsilon. \tag{1.2}$$

The equality case of inequality (1.2) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_m\}$ at p, such that the shape operator of M in $R^m(\varepsilon)$ at p with respect to $\{e_1, \ldots, e_m\}$ takes the form

$$A_r = \begin{pmatrix} A_1^r & & & \\ & \ddots & & \\ & & A_k^r & \\ & & & \mu_r I \end{pmatrix}, \quad r = n+1, \ldots, m, \tag{1.3}$$

where I is an identity matrix and A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying

$$\operatorname{tr} A_1^r = \cdots = \operatorname{tr} A_k^r = \mu_r, \tag{1.4}$$

and $c(n_1,\ldots,n_k)$ and $b(n_1,\ldots,n_k)$ are positive constants defined by

$$c(n_1,\ldots,n_k) = \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} \tag{1.5 - 1}$$

and

$$b(n_1,\ldots,n_k) = \frac{1}{2}(n(n-1)-\sum n_j(n_j-1)). \tag{1.5 - 2}$$

An isometric immersion in a real space form is called **ideal** in the sense of [3] if it satisfies the equality case of inequality (1.2) identically for some k-tuple, that is

$$\delta(n_1,\ldots,n_k) = c(n_1,\ldots,n_k)H^2 + b(n_1,\ldots,n_k)\varepsilon. \tag{1.6}$$

Ideal immersions associated with the simplest 1-tuple, namely $(2) \in \Psi(n)$, have been studied deeply in the last few years and many interesting results have been obtained (see for instance [1, 4, 5, 7]). But little is known concerning ideal immersions associated with a general k-tuple $(n_1,\ldots,n_k) \in \Psi(n)$ except those of immersions in a complex space form (see [2]). Dillen, Petrovic and Verstraelen (cf. [7]) have completely classified the ideal Einstein, ideal conformally flat and semi-symmetric immersions associated with $(2) \in \Psi(n)$. In this paper we consider such ideal immersion associated with the general k-tuple $(n_1,\ldots,n_k) \in \Psi(n)$. For convenience denote by $s_k = n_1 + \cdots + n_k$ (and sometimes write s).

First we have the following theorems about ideal Einstein and conformally flat immersions.

THEOREM 1: *Every ideal immersion in real space forms satisfying $n \geq s+1$ is Einstein if and only if it is totally geodesic.*

THEOREM 2: *Every ideal immersion in real space forms satisfying $n \geq s+2$ and $k \geq 2$ is conformally flat if and only if it is totally geodesic.*

A Riemannian manifold is called **semi-symmetric** (cf. [8]) if it satisfies $R\cdot R = 0$ (see section 2 below). The second purpose of this paper is classifying all ideal semi-symmetric hypersurfaces of real space forms.

THEOREM 3: *Let M^n be an ideal immersion in a Euclidean space E^{n+1}. Then M is semi-symmetric if and only if M is locally congruent to an open part of one of the following hypersurfaces:*

(1) *a totally geodesic hyperplane E^n,*

(2) *an $(n-2)$-ruled minimal hypersurface,*

(3) *a standard* n*-sphere* S^n,

(4) *an elliptic hypercone* C^n,

(5) *the product of a standard* l*-sphere* S^l *and an* $(n-l)$*-dimensional affine subspace* E^{n-l} *and* $l \geq 2$,

(6) *the product of an* l*-dimensional elliptic hypercone* C^l *and an* $(n-l)$*-dimensional affine subspace* E^{n-l} *and* $l \geq 2$,

where C^n *is the elliptic hypercone of* E^{n+1} *(for definition see [6]), and* S^l *and* C^l *are the hypersphere and hypercone in* E^{l+1} *respectively; and* E^{n-l} *is a Euclidean subspace of* E^{n+1} *orthogonal to* E^{l+1}.

THEOREM 4: *Every ideal immersed hypersurface* M *in a sphere* $S^{n+1}(1)$ *is semi-symmetric if and only if* M *is locally congruent to a hypersphere* $S^n(c)$ *or a Riemannian isoparametric torus* $S^l(a) \times S^{n-l}(b)$ *for suitable* a *and* b *where* $a^2 + b^2 = 1$ *and* $l = 1, \ldots, n-1$.

THEOREM 5: *Every ideal immersed hypersurface in a hyperbolic space* $H^{n+1}(-1)$ *is semi-symmetric if and only if* M *is locally congruent to an umbilical hypersurface or an isoparametric hypersurface* $S^l(a) \times H^{n-l}(b)$ *for suitable* a *and* b *satisfying* $a^2 - b^2 = -1$, *and* $l = 1, \ldots, n-1$.

2. Some simple lemmas

Let x: $M^n \to R^m(\varepsilon)$ be an isometric immersion of a real space form of constant curvature ε. For any k-tuple $(n_1, \ldots, n_k) \in \Psi(n)$, denote by I_i the index set $\{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\}$, $i = 1, \ldots, k$ and $s + 1 \leq \alpha, \beta, \ldots \leq n$ where $s = n_1 + \cdots + n_k$. The Riemannian curvature tensor R is defined by $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ where ∇ is the Levi-Civita connection of M. Denote by H, h and Ric the mean curvature, the second fundamental form, and Ricci tensor respectively; h^r_{ij} is the component of h.

If we choose e_{n+1} parallel to the mean curvature vector, Theorem A can be rewritten as

LEMMA 2.1: *Let* x: $M^n \to R^m(\varepsilon)$ *be an isometric immersion in a real space form of constant curvature* ε. *Then for each* k*-tuple* $(n_1, \ldots, n_k) \in \Psi(n)$, *we have* (1.2) *and the equality case of* (1.2) *holds if and only if there exists an orthonormal basis*

$\{e_1,\dots,e_m\}$ such that the shape operator of M takes the form

(2.1)
$$A_{n+1}=\begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_s & & & \\ & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{pmatrix},\quad A_t=\begin{pmatrix} A_1^t & & & & & \\ & \ddots & & & & \\ & & A_k^t & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

satisfying $\sum_{i\in I_i} a_i=\mu$, $\operatorname{tr} A_i^t=0$, for all $t=n+2,\dots,m$, and $i=1,\dots,k$.

LEMMA 2.2: *The sectional curvatures of M are*

$$K_{ij}=K(e_i\wedge e_j)=\varepsilon+a_ia_j+\sum_{t=n+2}^{m}\{h_{ii}^th_{jj}^t-(h_{ij}^t)^2\},\quad i,j\in I_i;$$
$$K_{ij}=K(e_i\wedge e_j)=\varepsilon+a_ia_j+\sum_{t=n+2}^{m}h_{ii}^th_{jj}^t,\quad i\in I_i,\quad j\in I_j,\quad I_i\neq I_j;$$
$$K_{i\alpha}=K(e_i\wedge e_\alpha)=\varepsilon+a_i\mu,\quad i\in I_i,\quad s+1\le\alpha\le n;$$
$$K_{\alpha\beta}=K(e_\alpha\wedge e_\beta)=\varepsilon+\mu^2.$$

Proof: The Lemma is an immediate result of the Gaussian equation and (2.1).

LEMMA 2.3: *The Ricci curvatures of M are*

$$\operatorname{Ric}(e_i,e_i)=(n-1)\varepsilon+(n-s+k)a_i\mu-\left\{a_i^2+\sum_{t=n+2,j\in I_i}^{m}(h_{ij}^t)^2\right\},\tag{2.2}$$

for $i\in I_i$, and

$$\operatorname{Ric}(e_\alpha,e_\alpha)=(n-1)\varepsilon+(n-s+k-1)\mu^2.\tag{2.3}$$

Proof: By Lemma 2.2 and (2.1) we have

$$\begin{aligned}\operatorname{Ric}(e_i,e_i)=&K_{i1}+\cdots+K_{is}+\sum_{\alpha=s+1}^{n}K_{i\alpha}\\ =&(n-1)\varepsilon+a_i\left\{\sum_{i\neq j\in I_i}a_j+(k-1)\mu+(n-s)\mu\right\}\\ &+\sum_{t=n+2}^{m}h_{ii}^t\left(\sum_{i\neq j\in I_i}h_{jj}^t-\sum_{i\neq j\in I_i}(h_{ij}^t)^2\right)\\ =&(n-1)\varepsilon+(n-s+k)a_i\mu-a_i^2-\sum_{t=n+2,j\in I_i}^{m}(h_{ij}^t)^2,\end{aligned}$$

which implies (2.2); (2.3) is similar.

A Riemannian manifold M is called **semi-symmetric** (cf. [8]) if

$$(R(X,Y)R)(U,V)W = 0$$

for all $X, Y, U, V, W \in TM$, where $R(X,Y)R$ is defined by

$$\begin{aligned}(R(X,Y)\cdot R)(U,V)W &= R(X,Y)(R(U,V)W) \\ -R(R(X,Y)U,V)W &- R(U,R(X,Y)V)W - R(U,V)(R(X,Y)W).\end{aligned} \tag{2.4}$$

Semi-symmetry is a proper generalization of local symmetry ($\nabla R = 0$).

The Gaussian equation of an isometric immersed hypersurface x: $M^n \to R^{n+1}(\varepsilon)$ in a real space form can be written as

$$R(X,Y)Z = <Y,Z> X - <X,Z> Y + <AY,Z> AX - <AX,Z> AY. \tag{2.5}$$

Now we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ such that $Ae_i = \lambda_i e_i$, $i = 1, \ldots, n$.

Lemma 2.4: *An isometric immersed hypersurface x: $M^n \to R^{n+1}(\varepsilon)$ is semi-symmetric if and only if*

$$\lambda_i(\lambda_j - \lambda_k)(\varepsilon + \lambda_j\lambda_k) = 0 \tag{2.6}$$

where i, j and k are distinct.

Proof: By a direct calculation we have by (2.4) and (2.5) that

$$(R(e_j,e_k)R)(e_i,e_k)e_j = -\lambda_i(\lambda_j - \lambda_k)(\varepsilon + \lambda_j\lambda_k)e_i,$$

which implies (2.6). The converse also holds because $(R(X,Y)R)(U,V)W \equiv 0$ if the distinct number of vectors X, Y, U, V and W is different from 3.

J. Deprez proved the following local classification theorem (in [8]) of hypersurfaces immersed in a Euclidean space E^{n+1}.

Lemma 2.5: *Let M^n be an n-dimensional Riemannian manifold which is isometrically immersed in E^{n+1}, such that $\operatorname{rank} A \geq 3$ at some point. Then M^n is semi-symmetric if and only if M^n is locally congruent to an open part of one of the following hypersurfaces:*

(1) *a hypersphere S^n,*

(2) *an elliptic hypercone C^n,*

(3) *the product of a standard l-dimensional sphere S^l and an $(n-l)$-dimensional affine subspace E^{n-l} and $l \geq 3$,*

(4) *the product of an l-dimensional hypercone C^l and an $(n-l)$-dimensional affine subspace E^{n-l} and $l \geq 3$,*

where C^l , S^l and E^{n-l} are similar to Theorem A.

3. Proof of Theorem 1

If M is an ideal Einstein immersion, then Ric $= \frac{\tau}{n} g$, where g is the metric tensor of M. So by (2.2) and (2.3) we have

$$(n-s+k)a_i\mu - \left\{a_i^2 + \sum_{t=n+2, j\in I_1}^{m} (h_{ij}^t)^2\right\} = (n-s+k-1)\mu^2.$$

By summation in I_1 one finds from (2.1)

$$(n-s+k)\mu^2 - n_1(n-s+k-1)\mu^2 = \sum_{i\in I_1} a_i^2 + \sum_{t=n+2, i,j\in I_1}^{m} (h_{ij}^t)^2. \tag{3.1}$$

Equation (3.1) holds for $n_2, \ldots, n_k$, too. So by summation we have

$$\sum_{i=1}^{s} a_i^2 + \sum_{t=n+2, i,j\in I_\sigma, 1\leq\sigma\leq k,}^{m} (h_{ij}^t)^2 = [(n-s+k)(k-s)+s]\mu^2. \tag{3.2}$$

Now we claim that $f(k,s) = (n-s+k)(k-s)+s = s^2-(n+2k-1)s+(k^2+nk) \leq 0$ for any $k \geq 1$ and $2 \leq s \leq n-1$.

By definition for any k-tuple $(n_1, \ldots, n_k) \in \Psi(n)$, $k=1$ when $s=2$ and $n \geq 3$. Moreover $2 \leq s \leq n-1$, and $k \leq [\frac{n-1}{2}]$. An easy calculation shows that

$$f(k,2) = f(1,2) = 3-n \leq 0,$$
$$f(k,n-1) = k(k+2-n) \leq k\Big(\frac{n-1}{2}+2-n\Big) = -\frac{k(n-3)}{2} \leq 0,$$

which implies $f(k,s) \leq 0$ by the property of quadric functions. By this claim, (3.2) indicates that M is totally geodesic. The converse holds trivially. This completes the proof of Theorem 1.

Remark: When $\varepsilon = 0$, $k=1$ and $n_1 = 2$, Theorem 1 is obtained by Dillen, Petrovic and Verstraelen in [7].

4. Ideal conformally flat immersion

The Weyl's conformal curvature tensor C of an n-dimensional Riemannian manifold M is defined by

$$\begin{aligned} C_{ijkl} =& R_{ijkl} - \frac{1}{n-2}\{R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{ik}R_{jl} - \delta_{il}R_{jk}\} \\ &+ \frac{\tau}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \end{aligned} \tag{4.1}$$

with respect to an orthonormal frame field, where $R_{ik} = \mathrm{Ric}(e_i, e_k)$. For dimension $n \geq 3$, M is conformally flat if and only if $C = 0$.

Next we give the

Proof of Theorem 2: For $i, j \in I_1$ we have by $C_{ijij} = 0$,

$$K_{ij} - \frac{1}{n-2}\{R_{ii} + R_{jj}\} + \frac{\tau}{(n-1)(n-2)} = 0.$$

By Lemmas 2.2 and 2.3 we get

$$\begin{aligned} K_{ij} &+ \frac{1}{n-2}\left\{a_i^2 + a_j^2 + \sum_{t,l\in I_1}[(h_{il}^t)^2 + (h_{jl}^t)^2]\right\} \\ &- \frac{n-s+k}{n-2}(a_i + a_j)\mu + \frac{\tau}{(n-1)(n-2)} - \frac{2(n-1)}{n-2}\varepsilon = 0. \end{aligned} \tag{4.2}$$

Similarly, using $C_{\alpha\beta\alpha\beta} = 0$ and Lemmas 2.2 and 2.3 we have

$$\varepsilon + \frac{\tau}{(n-1)(n-2)} = \frac{n+2k-2s}{n-2}\mu^2 + \frac{2(n-1)}{n-2}\varepsilon. \tag{4.3}$$

Inserting (4.3) into (4.2) we obtain

$$\begin{aligned} K_{ij} &+ \frac{1}{n-2}\left\{a_i^2 + a_j^2 + \sum_{t,l\in I_1}[(h_{il}^t)^2 + (h_{jl}^t)^2]\right\} \\ &- \frac{n-s+k}{n-2}(a_i + a_j)\mu + \frac{n+2k-2s}{n-2}\mu^2 - \varepsilon = 0. \end{aligned} \tag{4.4}$$

Also, $C_{i\alpha i\alpha} = 0$ together with Lemmas 2.2 and 2.3 gives

$$\begin{aligned} &\frac{s-k-2}{n-2}a_i\mu - \frac{n-s+k-1}{n-2}\mu^2 + \frac{1}{n-2}\left\{a_i^2 + \sum_{t,l\in I_1}(h_{il}^t)^2\right\} \\ &+ \frac{\tau}{(n-1)(n-2)} + \varepsilon - \frac{2(n-1)}{n-2}\varepsilon = 0. \end{aligned} \tag{4.5}$$

Inserting (4.3) into (4.5) we have

$$\frac{1}{n-2}\Big\{a_i^2+\sum_{t,l\in I_1}(h_{il}^t)^2\Big\}+\frac{s-k-2}{n-2}a_i\mu-\frac{s-k-1}{n-2}\mu^2=0. \tag{4.6}$$

For another $j \in I_1$, (4.6) also holds. By summation we get

$$\begin{aligned}&\frac{1}{n-2}\Big\{a_i^2+a_j^2+\sum_{t,l\in I_1}[(h_{il}^t)^2+(h_{jl}^t)^2]\Big\}\\ &+\frac{s-k-2}{n-2}(a_i+a_j)\mu-\frac{2(s-k-1)}{n-1}\mu^2=0.\end{aligned} \tag{4.7}$$

Combination of (4.4) and (4.7) gives

$$(a_i+a_j)\mu+\varepsilon-K_{ij}-\mu^2=0,\quad \forall i\neq j\in I_1. \tag{4.8}$$

(4.6) holds for all $i \in I_1$, so by summation in I_1 we get

$$\frac{1}{n-2}\Big\{\sum_{i\in I_1}a_i^2+\sum_{t,i,j\in I_1}(h_{ij}^t)^2\Big\}=\frac{n_1(s-k-1)-(s-k-2)}{n-2}\mu^2. \tag{4.9}$$

(4.9) also holds for any n_i, so by summation for n_i we have

$$\frac{1}{n-2}\Big\{\sum_{i=1}^{s}a_i^2+\sum_{t,i,j}(h_{ij}^t)^2\Big\}=\frac{s(s-k-1)-k(s-k-2)}{n-2}\mu^2,$$

which implies by (2.1) that

$$\|h\|^2=[n+(s-k)(s-k-2)]\mu^2. \tag{4.10}$$

Now (4.8) holds for any K_{ij} and $i \neq j$ in I_1; then by summation we have

$$\frac{1}{2}\tau(L_1)=\sum_{1\le i<j\le n_1}K_{ij}=\frac{n_1(n_1-1)}{2}\varepsilon+\frac{(n_1-1)(2-n_1)}{2}\mu^2.$$

The above equation also holds for any n_i, thus by summation again

$$\frac{1}{2}\sum_{i=1}^{k}\tau(L_i)=\frac{1}{2}\sum_{i=1}^{k}n_i(n_i-1)\varepsilon+\frac{1}{2}\{3s-2k-\sum n_i^2\}\mu^2. \tag{4.11}$$

Because M is an ideal immersion, then from (1.1), (1.5), (1.6) and (4.11) we know that

$$\begin{aligned}\tau=&\{(n^2-2ns+2nk-n)+(k^2+s^2-2ks+4s-3k-\sum n_i^2)\}\mu^2\\ &+n(n-1)\varepsilon.\end{aligned} \tag{4.12}$$

On the other hand, by the Gaussian equation and (4.10) we have

$$\tau = n(n-1)\varepsilon + [(n^2 - 2ns + 2nk - n) + (2s - 2k)]\mu^2. \tag{4.13}$$

Here we have used

$$H^2 = \frac{(n-s+k)^2}{n^2}\mu^2.$$

Combination of (4.12) and (4.13) gives

$$\left[(k-s)^2 + 2s - k - \sum_{i=1}^{k} n_i^2\right]\mu^2 = 0. \tag{4.14}$$

LEMMA 4.1: *Let $\omega(k) = (k-s)^2 + 2s - k - \sum_{i=1}^{k} n_i^2$. Then $\omega(k) > 0$ for all $k \geq 2$ and $n \geq s+2$.*

Proof (Using induction): First note that the Lemma holds for $k = 2$,

$$\omega(2) = 2(n_1 n_2 - n_1 - n_2 + 1) = 2(n_1 - 1)(n_2 - 1) > 0.$$

We assume the lemma holds for any k; then for $k+1$,

$$\begin{aligned}
\omega(k+1) =& (k+1-s_{k+1})^2 + 2s_{k+1} - (k+1) - \sum_{i=1}^{k+1} n_i^2 \\
=& [(k-s_k) - (n_{k+1}-1)]^2 + 2s_k + 2n_{k+1} - k - 1 - \sum_{i=1}^{k} n_i^2 - n_{k+1}^2 \\
=& [(k-s_k)^2 + 2s_k - k - \sum_{i=1}^{k} n_i^2] \\
&+ [(n_{k+1}-1)^2 - 2(k-s_k)(n_{k+1}-1) - (n_{k+1}^2 + 1 - 2n_{k+1})] \\
=& \omega(k) + 2(s_k - k)(n_{k+1} - 1) > 0.
\end{aligned}$$

Thus the lemma follows.

By Lemma 4.1, (4.14) implies that $\mu = 0$. Now by (4.10), $\|h\|^2 = 0$, i.e., M is totally geodesic. The converse is obvious. This completes the proof of Theorem 2.

From the proof we know that when $k = 1$, (4.13) becomes

$$\tau = (n-1)(n+2-2n_1)\mu^2 + n(n-1)\varepsilon.$$

Thus we have immediately the following corollaries.

COROLLARY 4.2: *Let $k = 1$, $n \geq n_1 + 2$ and $n \neq 2(n_1 - 1)$. Then when $\tau = n(n-1)\varepsilon$, M is ideal conformally flat if and only if M is totally geodesic.*

COROLLARY 4.3: *For $k = 1$, $n_1 = 2$, M is ideal conformally flat if and only if $K_{12} = \varepsilon$.*

Remark 4.4: When $k > 1$ and $s = n+1$, by a similar calculation as in the proof of Theorem 2 above we obtain

$$\tau = \frac{s}{s^2 - \sum_{i=1}^k n_i^2}[2s^2 - 2k\sum_{i=1}^k n_i^2 + (k^2+k-2)s - (k^2-k)]\mu^2 + s(s+1)\varepsilon.$$

Let $g(n_1, \ldots, n_k) = 2s^2 - 2k\sum_{i=1}^k n_i^2 + (k^2+k-2)s - (k^2-k) \neq 0$. It is easy to see that if $\tau = n(n-1)\varepsilon$, then M is ideal conformally flat if and only if M is totally geodesic.

5. Ideal semi-symmetric hypersurfaces

Proof of Theorem 3: If $\operatorname{rank} A \geq 3$ at some point, then Lemma 2.5 says that M is semi-symmetric if and only if M is one of (3)–(6) in Theorem 3, where $l \geq 3$ in (5) and (6). First note that any ideal immersion is one of the above 4 kinds of hypersurfaces if it is semi-symmetric. Secondly, the above 4 immersed hypersurfaces are all ideal immersions. Next we consider the case $\operatorname{rank} A \leq 2$.

If $\operatorname{rank} A = 0$, M is the totally geodesic hyperplane.

If $\operatorname{rank} A = 1$, the case doesn't happen because M is ideal.

If $\operatorname{rank} A = 2$, we assume $a_i \neq 0$, $a_j \neq 0$ and $i \neq j$. First note that we can assume $i, j \in I_1$. Otherwise, if $i \in I_1$ and $j \in I_2$, we have $a_i = a_j = \mu$. When μ is constant, M is case (5) for $l = 2$, and when $\mu \neq$ constant, M is case (6) for $l = 2$. Now for $i, j \in I_1$, $a_i + a_j = 0$. So M is an $(n-2)$-ruled minimal hypersurface. The converse also holds. This completes the proof of Theorem 3.

In order to prove Theorems 4 and 5, we give the following lemmas.

LEMMA 5.1: *Let $x\colon M^n \to R^{n+1}(\varepsilon \neq 0)$ be a minimal hypersurface in the space form with non-zero constant curvature ε. Then*

(1) when $\varepsilon = 1$, M is semi-symmetric if and only if M is totally geodesic or has two distinct principal curvatures

$$\lambda_1 = \sqrt{\frac{n-l}{l}} \quad \text{and} \quad \lambda_2 = -\sqrt{\frac{l}{n-l}}$$

with multiplicities l and $n-l$, respectively;

(2) *when $\varepsilon=-1$, M is semi-symmetric if and only if M is totally geodesic.*

Proof: If M is not totally geodesic, there exist two non-zero principal curvatures such that $\lambda_1 \neq \lambda_2$. By (2.6),

$$\lambda_1(\lambda_2-\lambda_i)(\varepsilon+\lambda_2\lambda_i)=0 \tag{5.1}$$

and

$$\lambda_2(\lambda_1-\lambda_i)(\varepsilon+\lambda_1\lambda_i)=0, \tag{5.2}$$

for any $i \neq 1,2$. Obviously λ_i is either λ_1 or λ_2. We assume that the multiplicities of λ_1 and λ_2 are l and $n-l$, respectively. Then

$$l\lambda_1+(n-l)\lambda_2=0. \tag{5.3}$$

Moreover, by (5.1) and (5.2) we have

$$\lambda_1\lambda_2+\varepsilon=0. \tag{5.4}$$

By (5.3) and (5.4) we have

$$\lambda_1^2=\frac{n-l}{l}\varepsilon, \quad \lambda_2^2=\frac{l}{n-l}\varepsilon.$$

Thus we have

$$\lambda_1=\sqrt{\frac{n-l}{l}} \quad \text{and} \quad \lambda_2=-\sqrt{\frac{l}{n-l}} \quad \text{for } \varepsilon=1.$$

For $\varepsilon=-1$ this case doesn't happen. Thus Lemma 5.1 follows.

LEMMA 5.2: *Let $x\colon M^n \to R^{n+1}(\varepsilon \neq 0)$ be non-minimal and non-umbilical semi-symmetric hypersurfaces. Then M is ideal if and only if M has two constant principal curvatures a_1 and a_2 such that $a_1a_2+\varepsilon=0$.*

Proof: Case (1), $n \geq s+1$.

By assumption $\mu \neq 0$. If there is some $a_i=0$ (for instance $a_1=0$), then by (2.6), $\mu(a_1-a_i)(\varepsilon+a_1a_i)=-\mu a_i\varepsilon=0$ for $i\neq 1$, which implies $a_i=0$ for all i. It contradicts $\mu\neq 0$. Now we have $a_i \neq 0$ for all $i \in \{1,\ldots,s\}$. Similar to the proof of Lemma 5.1, $a_i=\mu$ or $a_i=-\varepsilon/\mu$. Without loss of generality we assume that the multiplicity of eigenvalue μ in the submatrix A_1 is l_1. By $\operatorname{tr} A_1=\mu$, we have

$$(n_1-l_1)(-\varepsilon/\mu)+l_1\mu=\mu$$

which implies that $\mu^2 = (n_1 - l_1)/(l_1 - 1)$, Note that $l_1 \neq 1$. Hence

$$\mu = \sqrt{\frac{n_1 - l_1}{l_1 - 1}} \quad \text{and} \quad a_i = -\sqrt{\frac{l_1 - 1}{n_1 - l_1}} \quad \text{for } \varepsilon = 1, \quad l_1 > 1,$$

and

$$\mu = \sqrt{n_1} \quad \text{and} \quad a_i = 1/\sqrt{n_1} \quad \text{for } \varepsilon = -1, \quad l_1 = 0.$$

The discussion for $n_2, \ldots, n_k$ is similar and $n_1 = n_2 = \cdots = n_k$.

Case (2), $n = s$ and $k \geq 2$.

Similar to case (1) none of the principal curvatures is zero and only two, say a_1 and a_2, are different and satisfy

$$a_1 a_2 + \varepsilon = 0. \tag{5.5}$$

Now we consider the two submatrixes A_1 and A_2. Without loss of generality we assume A_1 and A_2 take the following forms:

$$A_1 = \begin{pmatrix} a_1 I_{l_1} & & \\ & \ddots & \\ & & a_2 I_{n_1 - l_1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_1 I_{l_2} & & \\ & \ddots & \\ & & a_2 I_{n_2 - l_2} \end{pmatrix},$$

for some integral functions l_1 and l_2. Then by assumption

$$\begin{aligned} l_1 a_1 + (n_1 - l_1) a_2 &= l_2 a_1 + (n_2 - l_2) a_2, \\ (l_1 - l_2) a_1 &= [(n_2 - n_1) + (l_1 - l_2)] a_2. \end{aligned} \tag{5.6}$$

Let $l = l_1 - l_2$. By (5.5) and (5.6) we get

$$l(-\varepsilon - a_1^2) = (n_2 - n_1)\varepsilon, \tag{5.7}$$

which implies that l is a constant integer because a_1 is a continuous function. If $l \neq 0$, it follows from (5.7) that a_1 is constant and so is a_2. If $l = 0$, then $l_1 = l_2$. A similar discussion applies to $n_3, \ldots, n_k$ and we conclude that the multiplicities of a_1 and a_2 are all $n/2$ where n is even in this case. Let $D(a_1) = \text{span}\{e_1, \ldots, e_{n/2}\}$ and $D(a_2) = \text{span}\{e_{\frac{n}{2}+1}, \ldots, e_n\}$ be the curvature distributions of a_1 and a_2, respectively. Obviously $n/2 > 1$. It's well known that the principal curvatures are constant along the integral manifolds of their corresponding distributions. So $e_i(a_1) = 0$ for $i \in \{1, \ldots, n/2\}$ and $e_j(a_2) = 0$ for $j \in \{n/2 + 1, \ldots, n\}$. Thus $0 = e_i(a_1 a_2 + \varepsilon) = e_i(a_1) a_2 + a_1 e_i(a_2) = a_1 e_i(a_2)$, which implies that $e_i(a_2) = 0$ for $i \in \{1, \ldots, n/2\}$ and so a_2 is constant. By (5.5), a_1 is constant, too. This completes the proof of the Lemma.

Proof of Theorems 4 and 5: By Lemmas 5.1 and 5.2, we know M is an isoparametric hypersurface with at most two distinct principal curvatures. Thus by a famous classification theorem of Cartan, if M is ideal semi-symmetric, it must be the hypersurfaces described in Theorems 4 and 5. Conversely, the above hypersurfaces are also ideal semi-symmetric immersions for suitable a and b. So Theorem 4 and Theorem 5 follows.

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